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Continuity of the optimal value function in indefinite quadratic programming

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Abstract. This paper characterizes the continuity property of the optimal value function in a general parametric quadratic programming problem with linear constraints. The lower semicontinuity and upper semicontinuity properties of the optimal value function are studied as well.

Key words: Quadratic programming problem, optimal value function, lower semicontinuity, upper semicontinuity, continuity

1. Introduction

Quadratic programming (QP) is a special branch of mathematical programming which has various theoretical and practical applications (see, for instance, [2, 3, 7–10, 13]).

Let R^n and R^m be finite-dimensional Euclidean spaces equipped with the standard scalar product and the Euclidean norm, $R^{m\times n}$ the space of $(m \times n)-$matrices equipped with the matrix norm induced by the vector norms in R^n and R^m. Let $R_S^{n\times n}$ be the space of symmetric $(n \times n)-$matrices equipped with the matrix norm induced by the vector norms in R^n. Let

$$\Omega := R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m.$$

Consider the following general quadratic programming problem with linear constraints, which will be denoted by $QP(D, A, c, b)$,

$$\begin{cases} \min f(x, c, D) := c^T x + \frac{1}{2} x^T D x \\ x \in \Delta(A, b) := \{x \in R^n : Ax \geqslant b\} \end{cases} \tag{1}$$

depending on the parameter $\omega = (D, A, c, b) \in \Omega$, where the superscript ^{T} denotes the transposition. The feasible region and the solution set of (1) will be denoted, respectively, by $\Delta(A, b)$ and $Sol(D, A, c, b)$. The function

$$\varphi : \Omega \longrightarrow R \cup \{\pm\infty\}$$

defined by

$$\varphi(\omega) = \begin{cases} \inf\{f(x, c, D) : x \in \Delta(A, b)\} & \text{if } \Delta(A, b) \neq \emptyset \\ +\infty & \text{if } \Delta(A, b) = \emptyset, \end{cases}$$

where $\omega = (D, A, c, b)$, is said to be the optimal value function of the parametric problem (1).

If $v^T D v \geqslant 0$ (resp., $v^T D v \leqslant 0$) for all $v \in R^n$ then $f(\cdot, c, D)$ is a convex (resp., concave) function and (1) is said to be a convex (resp., concave) QP problem. If such conditions are not required then we say that (1) is an indefinite QP problem.

In [4, 5] the authors have considered convex quadratic programming problems and obtained some results on the continuity and differentiability of the optimal value function of the problem as a function of a parameter specifying the magnitude of the perturbation. In [1], similar questions for the case of linear-quadratic programming problems were investigated. Continuity and Lipschitzian properties of the function $\varphi(D, A, \cdot, \cdot)$ (the matrices D and A are fixed) were studied in [2, 3, 9, 13]. Recently, various continuity properties of the Karush-Kuhn-Tucker point set and the solution map in indefinite quadratic programming problems have been investigated in [15–17].

In this paper, we consider indefinite quadratic programming problems and obtain several results on the continuity, the upper and lower semicontinuity of the optimal value function φ at a given point ω. The obtained results can be used for analyzing algorithms for solving the indefinite QP problems.

In Section 2, complete characterizations of the continuity of the function φ at a given point are obtained. In Section 3, sufficient conditions for the upper and lower semicontinuity of φ at a given point are established. Section 4 is devoted to some remarks related to the continuity and piecewise quadratic property of the function $\varphi(D, A, \cdot, \cdot)$.

In proofs we use the stability results of the feasible region $\Delta(A, b)$ due to Robinson [11, 12] and the Frank–Wolfe theorem on the existence of a global minimum in a QP problem (see [6, 7]).

Our results are compared with those obtained by Best and Ding [5] on the continuity of the function φ in convex QP problems (see Section 3).

The following notation will be adopted. The scalar product of vectors x, y and the Euclidean norm of a vector x in a finite-dimensional Euclidean space are denoted by $x^T y$ and $\|x\|$, respectively. Vectors in Euclidean spaces are interpreted as columns of real numbers. The notation $x \geqslant y$ (resp., $x > y$) means that every component of x is greater or equal (resp., greater) the corresponding component of y. For $A \in R^{m \times n}$, the matrix norm of A is given by

$$\|A\| = \max\{\|Ax\| : x \in R^n,\ \|x\| \leqslant 1\}.$$

For $D \in R_S^{n \times n}$, we define

$$\|D\| = \max\{\|Dx\| : x \in R^n,\ \|x\| \leqslant 1\}.$$

The norm in the product space $X_1 \times \cdots \times X_k$ of the normed spaces $X_1, \ldots, X_k$ is set to be

$$\|(x_1, \ldots, x_k)\| = \left(\|x_1\|^2 + \cdots + \|x_k\|^2\right)^{1/2}.$$

2. Continuity of the function $\varphi(\cdot)$

In this section we characterize the continuity of the function φ at a given point $\omega = (D, A, c, b)$. In comparison with the preceding results of Best and Chakravarti [4] and Best and Ding [5], the advantage here is that the quadratic objective function is allowed to be indefinite. Before obtaining the desired characterizations, we state some lemmas.

The following regularity condition (the Slater condition) for linear inequality systems is equivalent to the one in [11, p. 755]. Let $A \in R^{m\times n},\ b \in R^m$ be given. The system $Ax \geqslant b$ is said to be *regular* if there exists $x_0 \in R^n$ such that $Ax_0 > b$.

The following result is well-known (see, for example, [11, Theorem 1] and [2, Theorem 3.1.5]). For the completeness of the presentation, we provide a simple proof.

LEMMA 2.1. *Let $A \in R^{m\times n},\ b \in R^m$. The system $Ax \geqslant b$ is regular if and only if the set-valued map $\Delta(\cdot) : R^{m\times n} \times R^m \longrightarrow 2^{R^n}$, defined by $\Delta(A', b') = \{x \in R^n :\ A'x \geqslant b'\}$, is lower semicontinuous at (A, b). That is, $\Delta(A, b)$ is nonempty and, for every open set $V \subset R^n$ satisfying $\Delta(A, b) \cap V \neq \emptyset$ there exists $\delta > 0$ such that $\Delta(A', b') \cap V \neq \emptyset$ for every $(A', b') \in R^{m\times n} \times R^m$ with the property that $\|(A', b') - (A, b)\| < \delta$.*

Proof. Suppose that $Ax \geqslant b$ is a regular system and $x_0 \in R^n$ is such that $Ax_0 > b$. Obviously, $\Delta(A, b)$ is nonempty. Let V be an open subset in R^n satisfying $\Delta(A, b) \cap V \neq \emptyset$. Take $x \in \Delta(A, b) \cap V$. For every $t \in [0, 1]$, we set

$$x_t := (1 - t)x + tx_0.$$

Since $x_t \to x$ as $t \to 0$, there is $t_0 > 0$ such that $x_{t_0} \in V$. Since

$$Ax_{t_0} = (1 - t_0)Ax + t_0 Ax_0 > (1 - t_0)b + t_0 b = b,$$

there exists $\delta_{t_0} > 0$ such that

$$A'x_{t_0} > b'$$

for all $(A', b') \in R^{m\times n} \times R^m$ satisfying

$$\|(A', b') - (A, b)\| < \delta_{t_0}. \tag{2}$$

Thus, $x_t \in \Delta(A', b')$ for every (A', b') fulfilling (2). Therefore $\Delta(\cdot)$ is lower semicontinuous at (A, b).

Conversely, if $\Delta(\cdot)$ is lower semicontinuous at (A, b) then there exists $\delta > 0$ such that $Ax \geqslant b'$ is solvable for every $b' \in R^m$ satisfying $b' > b$ and $\|b' - b\| < \delta$. This implies that $Ax > b$ is solvable. Thus $Ax \geqslant b$ is a regular system. □

REMARK 2.1. If the inequality system $Ax \geqslant b$ is irregular then there exists a sequence $\{(A_k, b_k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A_k x \geqslant b_k$ has no solutions. This fact follows from the results of [12].

LEMMA 2.2. (cf. [12, Lemma 3]). *Let $A \in R^{m\times n}$. If the system $Ax \geqslant 0$ is regular then, for every $b \in R^m$, the system $Ax \geqslant b$ is regular.*

Proof. Assume that $Ax \geqslant 0$ is a regular and $\bar{x} \in R^n$ is such that $A\bar{x} > 0$. Setting $\bar{b} = A\bar{x}$, we have $\bar{b} > 0$. Let $b \in R^m$ be given arbitrarily. Then there exists $t > 0$ such that $t\bar{b} > b$. We have $A(t\bar{x}) = tA\bar{x} = t\bar{b}$. Therefore $A(t\bar{x}) > b$, hence the system $Ax \geqslant b$ is regular. □

LEMMA 2.3. *The set $\mathcal{G} := \{(D, A) \in R_s^{n\times n} \times R^{m\times n} : Sol(D, A, 0, 0) = \{0\}\}$ is open in $R_s^{n\times n} \times R^{m\times n}$.*

Proof. Suppose, contrary to our claim, that $\mathcal{G}$ is not open in $R_s^{n\times n} \times R^{m\times n}$. Then there exists a sequence $\{(D_k, A_k)\}$ in $R_s^{n\times n} \times R^{m\times n}$ converging to $(D, A) \in \mathcal{G}$ such that $Sol(D_k, A_k, 0, 0) \neq \{0\}$ for every k. Then, for every k, one can find $x_k \in R^n$ such that $\|x_k\| = 1$ and

$$A_k x_k \geqslant 0, \quad x_k^T D_k x_k \leqslant 0. \tag{3}$$

The sequence $\{x_k\}$ is bounded, hence it has a convergent subsequence. Without loss of generality, we may assume that the sequence $\{x_k\}$ itself converges to some $\bar{x} \in R^n$ with $\|\bar{x}\| = 1$. Taking the limits in the inequalities in (3) as $k \to \infty$, we obtain

$$A\bar{x} \geqslant 0, \quad \bar{x}^T D\bar{x} \leqslant 0.$$

This contradicts the assumption that $Sol(D, A, 0, 0) = \{0\}$. The proof is complete. □

LEMMA 2.4. *If $\Delta(A, b)$ is nonempty and if $Sol(D, A, 0, 0) = \{0\}$ then, for every $c \in R^n$, $Sol(D, A, c, b)$ is a nonempty compact set.*

Proof. Let $\Delta(A, b)$ be nonempty and $Sol(D, A, 0, 0) = \{0\}$. Suppose that $Sol(D, A, c, b) = \emptyset$ for some $c \in R^n$. By the Frank-Wolfe Theorem (see [6] and [7, Theorem 2.8.1]), there exists a sequence $\{x_k\}$ such that $Ax_k \geqslant b$ for every k and

$$f(x_k, c, D) = c^T x_k + \frac{1}{2} x_k^T D x_k \to -\infty \quad \text{as} \quad k \to \infty.$$

It is clear that $\|x_k\| \to +\infty$ as $k \to \infty$. By taking a subsequence, if necessary, we may assume that $\|x_k\|^{-1} x_k \to \bar{x} \in R^n$ and

$$f(x_k, c, D) = c^T x_k + \frac{1}{2} x_k^T D x_k < 0 \quad \text{for every } k. \tag{4}$$

We have

$$A \frac{x_k}{\|x_k\|} \geqslant \frac{b}{\|x_k\|}.$$

Letting $k \to \infty$, we obtain $\bar{x} \in \Delta(A, 0)$. Dividing both sides of the inequality in (4) by $\|x_k\|^2$ and letting $k \to \infty$, we get $\bar{x}^T D\bar{x} \leqslant 0$. Since $\|\bar{x}\| = 1$, $Sol(D, A, 0, 0) \neq \{0\}$, contradicting the assumption $Sol(D, A, 0, 0) = \{0\}$. Thus $Sol(D, A, c, b)$ is nonempty for each $c \in R^n$.

If, for some $c \in R^n$, the set $Sol(D, A, c, b)$ is unbounded then there exists a sequence $\{x_k\}$ such that $x_k \in Sol(D, A, c, b)$ for every k, $\|x_k\| \to \infty$ as $k \to \infty$ and $\{\|x_k\|^{-1}x_k\}$ converges to a certain $\bar{x} \in R^n$. Taking any $x \in \Delta(A, b)$, we have

$$c^T x_k + \frac{1}{2}x_k^T Dx_k \leqslant c^T x + \frac{1}{2}x^T Dx, \tag{5}$$

$$Ax_k \geqslant b. \tag{6}$$

Dividing both sides of (5) by $\|x_k\|^2$, both sides of (6) by $\|x_k\|$, and letting $k \to \infty$, we obtain

$$\bar{x}^T D\bar{x} \leqslant 0, \quad A\bar{x} \geqslant 0.$$

Thus $Sol(D, A, 0, 0) \neq \{0\}$, a contradiction. We have proved that, for every $c \in R^n$, the solution set $Sol(D, A, c, b)$ is bounded. Fixing any $\bar{x} \in Sol(D, A, c, b)$ one has

$$Sol(D, A, c, b) = \{x \in \Delta(A, b) : f(x, c, D) = f(\bar{x}, c, D)\}.$$

Hence $Sol(D, A, c, b)$ is a closed set and, therefore, $Sol(D, A, c, b)$ is a compact set. □

We are now in a position to state our first theorem on the continuity of the optimal value function φ. This theorem describes the set of two conditions which is necessary and sufficient for the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has a finite value.

THEOREM 2.1. *Let* $(D, A, c, b) \in R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$. *Assume that* $\varphi(D, A, c, b) \neq \pm\infty$. *Then, the optimal value function* $\varphi(\cdot)$ *is continuous at* (D, A, c, b) *if and only if the following two conditions are satisfied:*

(a) The system $Ax \geqslant b$ *is regular,*

(b) $Sol(D, A, 0, 0) = \{0\}$.

Proof. First, suppose that $\varphi(\cdot)$ is continuous at $\omega := (D, A, c, b)$ and $\varphi(\omega) \neq \pm\infty$. If (a) is violated then, by Remark 2.1, there exists a sequence $\{(A_k, b_k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A_k x \geqslant b_k$ has no solutions. Consider the sequence $\{(D, A_k, c, b_k)\}$ in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Since $\Delta(A_k, b_k) = \{x : A_k x \geqslant b_k\}$ is empty for every k,

$$\varphi(D, A_k, c, b_k) = +\infty \quad \text{for every } k.$$

On the other hand, as $\varphi(\cdot)$ is continuous at ω and $\{(D, A_k, c, b_k)\}$ converges to ω, we have

$$\lim_{k\to\infty} \varphi(D, A_k, c, b_k) = \varphi(D, A, c, b) \neq \pm\infty.$$

We have arrived at a contradiction. This shows that (a) is fulfilled.

Now we suppose that (b) fails to hold. Then there is a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geqslant 0, \quad \bar{x}^T D\bar{x} \leqslant 0. \tag{7}$$

Consider the sequence $\{(D_k, A, c, b)\}$, where $D_k := D - \frac{1}{k}E$, E is the unit matrix in $R^{n\times n}$. From the assumption $\varphi(\omega) \neq \pm\infty$, it follows that $\Delta(A, b)$ is nonempty. Since $\Delta(A, b) \neq \emptyset$, from (7) and from the inclusion $\Delta(A, b)+\Delta(A, 0) \subseteq \Delta(A, b)$, we conclude that $\Delta(A, b)$ is unbounded. For every k, it follows from (7) that

$$\bar{x}^T D_k\bar{x} = \bar{x}^T(D - \frac{1}{k}E)\bar{x} < 0.$$

Hence, for any x belonging to $\Delta(A, b)$ and for any $t > 0$, we have $x+t\bar{x} \in \Delta(A, b)$ and

$$\begin{aligned} f(x + t\bar{x}, c, D_k) = {} & c^T(x + t\bar{x}) \\ & + \frac{1}{2}(x + t\bar{x})^T D_k(x + t\bar{x}) \to -\infty \quad \text{as} \quad t \to \infty. \end{aligned}$$

This implies that, for all k, $Sol(D_k, A, c, b) = \emptyset$ and $\varphi(D_k, A, c, b) = -\infty$. We have arrived at a contradiction, because $\varphi(\cdot)$ is continuous at ω, $\{(D_k, A, c, b)\}$ converges to ω and $\varphi(\omega) \neq \pm\infty$. We have proved that (b) holds true.

From now on we assume that (a), (b) are satisfied and $\{(D_k, A_k, c_k, b_k)\}$ is an arbitrary sequence in Ω converging to ω. The assumption (a) and Lemma 2.1 yield the existence of a positive integer k_0 such that $\Delta(A_k, b_k) \neq \emptyset$ for every $k \geqslant k_0$. The assumption (b) and Lemma 2.3 imply that there exists a positive integer $k_1 \geqslant k_0$ such that $Sol(D_k, A_k, 0, 0) = \{0\}$ for every $k \geqslant k_1$. By Lemma 2.4, one has $Sol(D_k, A_k, c_k, b_k) \neq \emptyset$ for every $k \geqslant k_1$. Therefore, for every $k \geqslant k_1$, $\varphi(D_k, A_k, c_k, b_k)$ is a finite number. This means that, for every $k \geqslant k_1$, there exists $x_k \in R^n$ satisfying

$$\varphi(D_k, A_k, c_k, b_k) = c_k^T x_k + \frac{1}{2}x_k^T Dx_k, \tag{8}$$

$$A_k x_k \geqslant b_k. \tag{9}$$

By the Frank–Wolfe Theorem (see [6], [7, Theorem 2.8.1]), since $\varphi(\omega) \neq \pm\infty$,

$$Sol(D, A, c, b) \neq \emptyset.$$

Taking any $x_0 \in Sol(D, A, c, b)$, we have

$$\varphi(D, A, c, b) = c^T x_0 + \frac{1}{2}x_0^T Dx_0, \tag{10}$$

$$Ax_0 \geqslant b. \tag{11}$$

By Lemma 2.1, there exists a sequence $\{y_k\}$ in R^n converging to x_0 and

$$A_k y_k \geqslant b_k \quad \text{for every } k \geqslant k_1. \tag{12}$$

From (12) it follows that $y_k \in \Delta(A_k, b_k)$ for $k \geqslant k_1$. Then

$$\varphi(D_k, A_k, c_k, b_k) \leqslant c_k^T y_k + \frac{1}{2} y_k^T D_k y_k. \tag{13}$$

From (13) it follows that

$$\begin{aligned}\limsup_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) &\leqslant \limsup_{k\to\infty} (c_k^T y_k + \frac{1}{2} y_k^T D_k y_k) \\ &= \lim_{k\to\infty} (c_k^T y_k + \frac{1}{2} y_k^T D_k y_k).\end{aligned}$$

Therefore, taking account of (10) and (11), we get

$$\limsup_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \leqslant \varphi(D, A, c, b). \tag{14}$$

We now claim that the sequence $\{x_k\}$, $k \geqslant k_1$, is bounded. Indeed, if the sequence $\{x_k\}$, $k \geqslant k_1$, is unbounded then, by taking a subsequence if necessary, we may assume that $\|x_k\| \to \infty$ as $k \to \infty$ and $\|x_k\| \neq 0$ for all $k \geqslant k_1$. Then $\{\|x_k\|^{-1} x_k\}$, $k \geqslant k_1$, is a bounded sequence, hence it has a convergent subsequence. Without loss of generality, we may assume that the sequence $\{\|x_k\|^{-1} x_k\}$, $k \geqslant k_1$, converges to some $\hat{x} \in R^n$ with $\|\hat{x}\| = 1$. From (9) we have

$$A_k \frac{x_k}{\|x_k\|} \geqslant \frac{b_k}{\|x_k\|}.$$

Letting $k \to \infty$, we obtain

$$A\hat{x} \geqslant 0. \tag{15}$$

By (8) and (13),

$$c_k^T x_k + \frac{1}{2} x_k^T D_k x_k \leqslant c_k^T y_k + \frac{1}{2} y_k^T D_k y_k. \tag{16}$$

Dividing both sides of (16) by $\|x_k\|^2$ and taking limits as $k \to \infty$, we get

$$\hat{x}^T D \hat{x} \leqslant 0. \tag{17}$$

By (15) and (17), we have $Sol(D, A, 0, 0) \neq \{0\}$. This contradicts (b). We have thus shown that the sequence $\{x_k\}$, $k \geqslant k_1$, is bounded; hence it has a convergent sequence. Without loss of generality, we may assume that the sequence $\{x_k\}$, $k \geqslant k_1$, converges to a point $\tilde{x} \in R^n$. By (8) and (9),

$$\lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = c^T \tilde{x} + \frac{1}{2} \tilde{x}^T D \tilde{x} = f(\tilde{x}, c, D), \tag{18}$$

$$A\tilde{x} \geqslant b. \tag{19}$$

From (19) it follows that $\tilde{x} \in \Delta(A, b)$. Hence

$$f(\tilde{x}, c, D) \geqslant \varphi(D, A, c, b).$$

Therefore,

$$\lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \geqslant \varphi(D, A, c, b). \tag{20}$$

Combining (14) and (20) gives

$$\lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = \varphi(D, A, c, b).$$

This shows that φ is continuous at (D, A, c, b). The proof is complete. □

EXAMPLE 2.1. Consider the problem $QP(D, A, c, b)$ where $m = 3$, $n = 2$,

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}, \quad c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be verified that $\varphi(D, A, c, b) = 0$, $Sol(D, A, 0, 0) = \{0\}$, and the system $Ax \geqslant b$ is regular. By Theorem 2.1, φ is continuous at (D, A, c, b).

EXAMPLE 2.2. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [1]$, $A = [0]$, $c = (1)$, $b = (0)$. It can be shown that $\varphi(D, A, c, b) = 0$, and the system $Ax \geqslant b$ is irregular. By Theorem 2.1, φ is not continuous at (D, A, c, b).

REMARK 2.2. If $\Delta(A, b)$ is nonemty then $\Delta(A, 0)$ is the recession cone of $\Delta(A, b)$. By definition, $Sol(D, A, 0, 0)$ is the solution set of the problem $QP(D, A, 0, 0)$. So, verifying the assumption $Sol(D, A, 0, 0) = \{0\}$ is equivalent to solving one special QP problem. Note that this assumption is equivalent to the requirement that $x^T Dx > 0$ for all $x \in R^n \setminus \{0\}$ satisfying $Ax \geqslant 0$. In particular, $Sol(D, A, 0, 0) = \{0\}$ in the case where D is a positive definite matrix and in the case where $\Delta(A, b)$ is a bounded set (in the latter case, $\Delta(A, 0) = \{0\}$). It can be shown by examples that the assumption $Sol(D, A, 0, 0) = \{0\}$ is fulfilled for many other QP problems, and there are many QP problems where this assumption fails to hold.

Now we study the continuity of the optimal value function $\varphi(\cdot)$ at a point where its value is infinity. Let $\alpha \in \{+\infty, -\infty\}$ and $\varphi(D, A, c, b) = \alpha$. We say that $\varphi(\cdot)$ is continuous at (D, A, c, b) if, for every sequence $\{(D_k, A_k, c_k, b_k)\}$ in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ converging to (D, A, c, b),

$$\lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = \alpha.$$

The next theorem characterizes the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has the value $-\infty$.

THEOREM 2.2. *Let* $(D, A, c, b) \in R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ *and* $\varphi(D, A, c, b) = -\infty$. *Then, the optimal value function* φ *is continuous at* (D, A, c, b) *if and only if the system* $Ax \geqslant b$ *is regular.*

Proof. Suppose that $\varphi(D, A, c, d) = -\infty$ and φ is continuous at (D, A, c, b) but the system $Ax \geqslant b$ is irregular. By Remark 2.1, there exists a sequence $\{(A_k, b_k)\}$ in $R^{m\times n} \times R^m$ converging to (A, b) such that, for every k, the system $A_k x \geqslant b_k$ has no solutions. Consider the sequence $\{(D, A_k, c, b_k)\}$ in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Since $\Delta(A_k, b_k) = \emptyset$ for every k, $\varphi(D, A_k, c, b_k) = +\infty$ for every k. Therefore, $\lim_{k\to\infty} \varphi(D, A_k, c, b_k) = +\infty$. On the other hand, since φ is continuous at (D, A, c, b) and since $\{(D, A_k, c, b_k)\}$ converges to (D, A, c, b),

$$+\infty = \lim_{k\to\infty} \varphi(D, A_k, c, b_k) = \varphi(D, A, c, b) = -\infty.$$

We have arrived at a contradiction. This proves that $Ax \geqslant b$ is a regular system.

Conversely, assume that $\varphi(D, A, c, d) = -\infty$ and that the system $Ax \geqslant b$ is regular. Let $\{(D_k, A_k, c_k, b_k)\}$ be an arbitrary sequence in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ converging to (D, A, c, b). By the assumption, $\varphi(D, A, c, b) = -\infty$, hence there is a sequence $\{x^{(i)}\}$ in R^n such that

$$\begin{aligned} &f(x^{(i)}, c, D) = c^T x^{(i)} + \frac{1}{2}(x^{(i)})^T D x^{(i)}, \\ &Ax^{(i)} \geqslant b \end{aligned}$$

and

$$f(x^{(i)}, c, D) \longrightarrow -\infty \quad \text{as} \quad i \to \infty. \tag{21}$$

By Lemma 2.1, for every i, there exists a sequence $\{y_k^{(i)}\}$ in R^n with the property that

$$A_k y_k^{(i)} \geqslant b_k, \tag{22}$$

$$\lim_{k\to\infty} y_k^{(i)} = x^{(i)}. \tag{23}$$

By (22),

$$\varphi(D_k, A_k, c_k, b_k) \leqslant c_k^T y_k^{(i)} + \frac{1}{2}(y_k^{(i)})^T D_k y_k^{(i)}. \tag{24}$$

From (23) and (24) it follows that

$$\limsup_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \leqslant c^T x^{(i)} + \frac{1}{2}(x^{(i)})^T D x^{(i)}. \tag{25}$$

Combining (25) with (21), we obtain

$$\limsup_{k\to\infty} \varphi(D_k, A_k, c_, b_k) = -\infty.$$

This implies that

$$\lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = -\infty = \varphi(D, A, c, b).$$

Thus φ is continuous at (D, A, c, b). The proof is complete. □

The following theorem characterizes the continuity of φ at a point $\omega = (D, A, c, b)$ where φ has the value $+\infty$.

THEOREM 2.3. *Let $(D, A, c, b) \in R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ and $\varphi(D, A, c, b) = +\infty$. Then, the optimal value function φ is continuous at (D, A, c, b) if and only if $Sol(D, A, 0, 0) = \{0\}$.*

Proof. Suppose that $\varphi(D, A, c, b) = +\infty$ and that φ is continuous at (D, A, c, b) but $Sol(D, A, 0, 0) \neq \{0\}$. Then there exists a nonzero vector $\bar{x} \in R^n$ such that

$$A\bar{x} \geqslant 0, \quad \bar{x}^T D\bar{x} \leqslant 0.$$

Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)^T$. We define a matrix M of the order $m \times n$ by setting $M = [m_{ij}]$, where

$$m_{ij} = \bar{x}_j \quad \text{for } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n.$$

Let

$$D_k = D - \frac{1}{k}E, \quad A_k = A + \frac{1}{k}M,$$

where E is the unit matrix in $R^{n\times n}$. Consider the sequence $\{(D_k, A_k, c, b)\}$. A simple computation shows that

$$A_k\bar{x} > 0 \quad \text{for every } k.$$

By Lemma 2.2, for every k we have that the system $A_k x \geqslant b$ is regular, hence it is solvable. Let z be a solution of the system $A_k x \geqslant b$. Since $A_k\bar{x} > 0$ and

$$\bar{x}^T D_k\bar{x} = \bar{x}^T D\bar{x} - \frac{\bar{x}^T\bar{x}}{k} < 0$$

for every k,

$$f(z + t\bar{x}, c, D_k) = c^T(z + t\bar{x}) + \frac{1}{2}(z + t\bar{x})^T D_k(z + t\bar{x}) \to -\infty$$

as $t \to \infty$. Since $z + t\bar{x} \in \Delta(A_k, b)$ for every k and for every $t > 0$, $Sol(D_k, A_k, c, b) = \emptyset$. We have arrived at a contradiction, because φ is continuous at (D, A, c, b) and

$$-\infty = \lim_{k\to\infty} \varphi(D_k, A_k, c, b) = \varphi(D, A, c, b) = +\infty.$$

Conversely, assume that $Sol(D, A, 0, 0) = \{0\}$ and $\{(D_k, A_k, c_k, b_k)\}$ is an arbitrary sequence in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ converging to (D, A, c, b). We shall show that

$$\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = +\infty.$$

Suppose that $\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) < +\infty$. Without loss of generality, we may assume that

$$\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = \lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) < +\infty.$$

Then, there exist a positive integer k_1 and a constant $\gamma \geqslant 0$ such that

$$\varphi(D_k, A_k, c_k, b_k) \leqslant \gamma$$

for every $k \geqslant k_1$. As $Sol(D, A, 0, 0) = \{0\}$, by Lemma 2.3 we may assume that there is an positive integer k_2 such that $Sol(D_k, A_k, 0, 0) = \{0\}$ for every $k \geqslant k_2$. By Lemma 2.4 we may assume that $Sol(D_k, A_k, c_k, b_k) \neq \emptyset$ for every $k \geqslant k_2$. Hence there exists a sequence $\{x_k\}$ in R^n such that, for every $k \geqslant k_2$, we have

$$\varphi(D_k, A_k, c_k, b_k) = c_k^T x_k + \frac{1}{2} x_k^T D_k x_k \leqslant \gamma, \tag{26}$$

$$A_k x_k \geqslant b_k. \tag{27}$$

We now prove that $\{x_k\}$ is a bounded sequence. Suppose, contrary to our claim, that the sequence $\{x_k\}$ is unbounded. Without loss of generality, we may assume that $\|x_k\| \neq 0$ for every k and that $\|x_k\| \to \infty$ as $k \to \infty$. Then the sequence $\{\|x_k\|^{-1} x_k\}$ is bounded and it has a convergent subsequence. We may assume that the sequence $\{\|x_k\|^{-1} x_k\}$ itself converges to a point $x_0 \in R^n$ with $\|x_0\| = 1$. By (27) we have

$$A_k \frac{x_k}{\|x_k\|} \geqslant \frac{b_k}{\|x_k\|},$$

hence

$$Ax_0 \geqslant 0. \tag{28}$$

By dividing both sides of the inequality in (26) by $\|x_k\|^2$ and taking the limits as $k \to \infty$, we get

$$x_0^T D x_0 \leqslant 0. \tag{29}$$

By (28) and (29), we have $Sol(D, A, 0, 0) \neq \{0\}$, a contradiction to our assumption. Thus the sequence $\{x_k\}$ is bounded, and it has a convergent subsequence. Without loss of generality, we may assume that $\{x_k\}$ converges to $\bar{x} \in R^n$. Letting $k \to \infty$, from (27) we obtain

$$A\bar{x} \geqslant b.$$

This means that $\Delta(A, b) \neq \emptyset$. We have arrived at a contradiction because $\varphi(D, A, c, b) = +\infty$. The proof is complete. $\square$

From Theorems 2.1–2.3 it follows that conditions (a), (b) in Theorem 2.1 are sufficient for the function $\varphi(\cdot)$ to be continuous at the given parameter value (D, A, c, b).

In the next section we shall obtain sufficient conditions for the upper semicontinuity (resp., lower semicontinuity) of the function $\varphi(\cdot)$ at a given parameter value.

3. Semicontinuity of the function $\varphi(\cdot)$

As it has been shown in the preceding section, continuity of the optimal value function holds under a special set of conditions. In some situations, only the upper semicontinuity or the lower semicontinuity of that function is required. So one may wish to have simple sufficient conditions for the upper semicontinuity and the lower semicontinuity of φ at a given point. Such conditions are given in this section.

A sufficient condition for the upper semicontinuity of the function $\varphi(\cdot)$ at a given parameter value is given in the following theorem.

THEOREM 3.1. *Let $(D, A, c, b) \in R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$. If the system $Ax \geqslant b$ is regular then $\varphi(\cdot)$ is upper semicontinuous at (D, A, c, b).*

Proof. As $Ax \geqslant b$ is regular, we have $\Delta(A, b) \neq \emptyset$, hence

$$\varphi(D, A, c, b) < +\infty.$$

Let $\{(D_k, A_k, c_k, b_k)\}$ be an arbitrary sequence in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ converging to (D, A, c, b). Since $\varphi(D, A, c, b) < +\infty$, there is a sequence $\{x^{(i)}\}$ in R^n such that $Ax^{(i)} \geqslant b$ and

$$f\left(x^{(i)}, c, D\right) = c^T x^{(i)} + \frac{1}{2}\left(x^{(i)}\right)^T Dx^{(i)} \longrightarrow \varphi(D, A, c, b) \quad \text{as } i \to \infty.$$

By Lemma 2.1 and by the regularity of the system $Ax \geqslant b$, for each i one can find a sequence $\{y_k^{(i)}\}$ in R^n such that $A_k y_k^{(i)} \geqslant b_k$ and

$$\lim_{k\to\infty} y_k^{(i)} = x^{(i)}.$$

Since $y_k^{(i)} \in \Delta(A_k, b_k)$,

$$\varphi(D_k, A_k, c_k, b_k) \leqslant f(y^{(i)}, c_k, D_k).$$

This implies that

$$\limsup_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \leqslant f(x^{(i)}, c, D).$$

Taking limits in the last inequality as $i \to \infty$, we obtain

$$\limsup_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \leqslant \varphi(D, A, c, b).$$

We have proved that $\varphi(\cdot)$ is upper semicontinuous at (D, A, c, b). □

The next example shows that the regularity condition in Theorem 3.1 does not guarantee the lower semicontinuity of φ at (D, A, c, b).

EXAMPLE 3.1. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [0]$, $A = [1]$, $c = (0)$, $b = (0)$. It is clear that $Ax \geqslant 0$ is regular, $Sol(D, A, c, b) = \Delta(A, b) = \{x : x \geqslant 0\}$, and $\varphi(D, A, c, b) = 0$. Consider the sequence $\{(D_k, A, c, b)\}$, where $D_k = D - \left[\frac{1}{k}\right]$. We have $\varphi(D_k, A, c, b) = -\infty$ for every k, so

$$\liminf_{k\to\infty} \varphi(D_k, A, c, b) < \varphi(D, A, c, b).$$

Thus φ is not lower semicontinuous at (D, A, c, b).

The following example is designed to show that the regularity condition in Theorem 3.1 is sufficient but not necessary for the upper semicontinuity of φ at (D, A, c, b).

EXAMPLE 3.2. Choose a matrix $A \in R^{m\times n}$ and a vector $b \in R^m$ such that $\Delta(A, b) = \emptyset$ (then the system $Ax \geqslant b$ is irregular). Fix an arbitrary matrix $D \in R_S^{n\times n}$ and an arbitrary vector $c \in R^n$. Since $\varphi(D, A, c, b) = +\infty$, for any sequence $\{(D_k, A_k, c_k, b_k)\}$ converging to (D, A, c, b), we have

$$\limsup_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \leqslant \varphi(D, A, c, b).$$

Thus φ is upper semicontinuous at (D, A, c, b).

A sufficient condition for the lower semicontinuity of the function $\varphi(\cdot)$ is given in the following theorem.

THEOREM 3.2. *Let* $(D, A, c, b) \in R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$. *If* $Sol(D, A, 0, 0) = \{0\}$ *then* $\varphi(\cdot)$ *is lower semicontinuous at* (D, A, c, b).

Proof. Assume that $Sol(D, A, 0, 0) = \{0\}$. Let $\{(D_k, A_k, c_k, b_k)\}$ be an arbitrary sequence in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ converging to (D, A, c, b). We claim that

$$\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \geqslant \varphi(D, A, c, b).$$

Indeed, suppose that

$$\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) < \varphi(D, A, c, b).$$

Without loss of generality, we may assume that

$$\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) = \lim_{k\to\infty} \varphi(D_k, A_k, c_k, b_k).$$

Then there exist an index k_1 and a real number γ such that $\gamma < \varphi(D, A, c, b)$ and

$$\varphi(D_k, A_k, c_k, b_k) \leqslant \gamma \quad \text{for every } k \geqslant k_1.$$

Since $\varphi(D_k, A_k, c_k, b_k) < +\infty$, we must have $\Delta(A_k, b_k) \neq \emptyset$ for every $k \geqslant k_1$. As

$$Sol(D, A, 0, 0) = \{0\},$$

by Lemma 2.3 there exists an index $k_2 \geqslant k_1$ such that $Sol(D_k, A_k, 0, 0) = \{0\}$ for every $k \geqslant k_2$. As $\Delta(A_k, b_k) \neq \emptyset$, applying Lemma 2.4 we have $Sol(D_k, A_k, c_k, b_k) \neq \emptyset$ for every $k \geqslant k_2$. Hence there exists a sequence $\{x_k\}$ such that we have $A_k x_k \geqslant b_k$ for every $k \geqslant k_2$, and

$$c_k^T x_k + \frac{1}{2} x_k^T D_k x_k = \varphi(D_k, A_k, c_k, b_k) \leqslant \gamma.$$

The sequence $\{x_k\}$ must be bounded. Indeed, if $\{x_k\}$ is unbounded then, without loss of generality, we may assume that $\|x_k\| \neq 0$ for every k and $\|x_k\| \to \infty$ as $k \to \infty$. Then the sequence $\{\|x_k\|^{-1} x_k\}$ is bounded and, therefore, it has a convergent subsequence. We may assume that this sequence itself converges to a vector $v \in R^n$ with $\|v\| = 1$. Since

$$A_k \frac{x_k}{\|x_k\|} \geqslant \frac{b_k}{\|x_k\|} \quad \text{for every } k \geqslant k_2,$$

we have $Av \geqslant 0$. On the other hand, since

$$c_k^T \frac{x_k}{\|x_k\|} + \frac{1}{2} \frac{x_k^T}{\|x_k\|} D_k \frac{x_k}{\|x_k\|} \leqslant \frac{\gamma}{\|x_k\|^2}$$

then

$$v^T D v \leqslant 0.$$

Combining all the above we get $v \in Sol(D, A, 0, 0) \setminus \{0\}$, a contradiction. We have thus proved that the sequence $\{x_k\}$ is bounded. Without loss of generality, we may assume that $x_k \to \bar{x} \in R^n$. Since $A_k x_k \geqslant b_k$ for every k, we have $A\bar{x} \geqslant b$, that is $\bar{x} \in \Delta(A, b)$. Since

$$c_k^T x_k + \frac{1}{2} x_k^T D_k x_k \leqslant \gamma$$

then

$$f(\bar{x}, c, D) = c^T \bar{x} + \frac{1}{2} \bar{x}^T D \bar{x} \leqslant \gamma.$$

As $\gamma < \varphi(D, A, c, b)$, we have $f(\bar{x}, c, D) < \varphi(D, A, c, b)$. This is an absurd because $\bar{x} \in \Delta(A, b)$.

We have proved that $\varphi(\cdot)$ is lower semicontinuous at (D, A, c, b). $\square$

The next example shows that the condition $Sol(D, A, 0, 0) = \{0\}$ in Theorem 3.2 does not guarantee the upper semicontinuity of φ at (D, A, c, b).

EXAMPLE 3.3. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [1]$, $A = [0]$, $c = (0), b = (0)$. It is clear that $Sol(D, A, 0, 0) = \{0\}$. Consider the sequence $\{(D, A, c, b_k)\}$, where $b_k = (\frac{1}{k})$. We have $\varphi(D, A, c, b) = 0$ and $\varphi(D, A, c, b_k) = +\infty$ for all k (because $\Delta(A, b_k) = \emptyset$ for all k). Therefore

$$\limsup_{k\to\infty} \varphi(D, A, c, b_k) = +\infty > 0 = \varphi(D, A, c, b).$$

Thus φ is not upper semicontinuous at (D, A, c, b).

The condition $Sol(D, A, 0, 0) = \{0\}$ in Theorem 3.2 is sufficient but not necessary for the lower semicontinuity of φ at (D, A, c, b).

EXAMPLE 3.4. Consider the problem $QP(D, A, c, b)$ where $m = n = 1$, $D = [-1]$, $A = [1]$, $c = (1), b = (0)$. It is clear that $Sol(D, A, 0, 0) = \emptyset$. Since $\varphi(D, A, c, b) = -\infty$, for any sequence $\{(D_k, A_k, c_k, b_k)\}$ converging to (D, A, c, b), we have

$$\liminf_{k\to\infty} \varphi(D_k, A_k, c_k, b_k) \geqslant \varphi(D, A, c, b).$$

Thus φ is lower semicontinuous at (D, A, c, b).

In the remainder of this section we shall compare the results above with those obtained by Best and Ding [5] on the continuity of the function $\varphi(\cdot)$ in convex quadratic programming problems.

As in [5], we consider the QP problem $QP(D, A, c, b)$:

$$\begin{cases} \min f(x, c, D) := c^T x + \frac{1}{2} x^T D x \\ x \in \Delta(A, b) := \{x \in R^n : Ax \geqslant b\}, \end{cases}$$

and assume that D is a symmetric positive semidefinite matrix. Under that assumption, the function $f(\cdot, c, D)$ is convex. Together with the problem $QP(D, A, c, b)$ above, we consider the following parametric quadratic programming problem $QP(D(t), A(t), c(t), b(t))$:

$$\begin{cases} \min f(x, c(t), D(t)) := c^T(t)x + \frac{1}{2} x^T D(t)x \\ x \in \Delta(A(t), b(t)) := \{x \in R^n : A(t)x \geqslant b(t)\}, \end{cases}$$

where $t \in R^k$ is a parameter. Suppose that $D(t),\ A(t),\ c(t),\ b(t)$ are vector-valued functions which are continuous at $t = 0$, and $D(0) = D,\ A(0) = A,\ c(0) = c,\ b(0) = b$. Besides, for every t, $D(t)$ is a symmetric positive semidefinite matrix. Set

$$\varphi(t) = \inf\{f(x, c(t), D(t)) : x \in \Delta(A(t), b(t))\}.$$

(If $\Delta(A(t), b(t)) = \emptyset$ then we set $\varphi(t) = +\infty$). According to [5], $QP(D, A, c, b)$ is said to be a regular problem if the following two conditions are satisfied:

$(R1)$ There does not exist $s \in R^n \setminus \{0\}$ such that

$$As \geqslant 0, \quad c^T s \leqslant 0, \quad Ds = 0.$$

$(R2)$ There does not exist $\lambda \in R^m \setminus \{0\}$ such that

$$A^T \lambda = 0, \quad \lambda \geqslant 0, \quad b^T \lambda \geqslant 0.$$

The main results of [5] are stated as follows.

THEOREM 3.3. ([5, Theorem 2.1]). *If condition $(R1)$ holds then the function $\varphi(t)$ is lower semicontinuous at $t = 0$.*

THEOREM 3.4. ([5, Theorem 2.2]). *If the conditions $(R1)$ and $(R2)$ hold then the function $\varphi(t)$ is continuous at $t = 0$.*

Note that if $Sol(D, A, 0, 0) = \{0\}$ then $(R1)$ holds. Indeed, if $(R1)$ does not hold then there exists $\bar{s} \in R^n \setminus \{0\}$ such that $A\bar{s} \geqslant 0$, $c^T \bar{s} \leqslant 0$, $D\bar{s} = 0$. Therefore,

$$f(\bar{s}, 0, D) = \frac{1}{2}\bar{s}^T D\bar{s} = 0.$$

Hence $Sol(D, A, 0, 0) \neq \{0\}$.

From the remark above we conclude that the assumption of Theorem 3.2 is stronger than the assumption of Theorem 3.3. But we have to stress that Theorem 3.3 can be applied only to the class of convex problems, while Theorem 3.2 can be used also for nonconvex problems.

It can be shown that $(R2)$ is equivalent to the regularity of the inequality system $Ax \geqslant b$. Indeed, if the system $Ax \geqslant b$ is irregular (that is the system $Ax > b$ has no solutions) then for any sequence $\{b_k\}$ satisfying $b_k > b$ for all k, $b_k \to b$, the inequality systems $Ax \geqslant b_k$ have no solutions. Then, for each k there exists ([7, Theorem 2.7.8]) $\lambda_k \in R^m$ such that

$$A^T \lambda_k = 0, \quad \lambda_k \geqslant 0, \quad b_k^T \lambda_k > 0.$$

Since $\lambda_k \neq 0$, we have

$$A^T \frac{\lambda_k}{\|\lambda_k\|} = 0, \quad \frac{\lambda_k}{\|\lambda_k\|} \geqslant 0, \quad b_k^T \frac{\lambda_k}{\|\lambda_k\|} > 0.$$

This yields the existence of a vector $v \in R^n \setminus \{0\}$ satisfying

$$A^T v = 0, \quad v \geqslant 0, \quad b^T v \geqslant 0,$$

hence $(R2)$ is violated. Conversely, suppose that $(R2)$ does not hold, that is there exists $\lambda \in R^m \setminus \{0\}$ such that

$$A^T\lambda = 0, \quad \lambda \geqslant 0, \quad b^T\lambda \geqslant 0.$$

If there is $\bar{x}$ satisfying $A\bar{x} > b$, then

$$0 < (A\bar{x} - b)^T\lambda = \bar{x}^T A^T\lambda - b^T\lambda = -b^T\lambda \leqslant 0,$$

a contradiction. Therefore, the system $Ax \geqslant b$ is irregular.

The just mentioned equivalence and Theorem 3.1 show that if $(R2)$ is valid then $\varphi(t)$ is upper semicontinuous at $t = 0$. Thus, Theorem 3.4 is a direct corollary of Theorems 3.1 and 3.3.

The following example illustrates the results presented in this section.

EXAMPLE 3.5. Consider the problem $QP(D, A, c, b)$ where $n = m = 1$, $D = [1]$, $A = [1]$, $c = (-1)$, $b = (0)$. Direct calculation shows that $Sol(D, A, 0, 0) = \{0\}$, and that conditions $(R1)$, $(R2)$ hold. (Then the system $Ax \geqslant b$ is regular.) Thus the assumptions of Theorems 3.1–3.4 are satisfied. Note that, in this case, the assertions of Theorems 3.1 and 3.2 cover those of Theorems 3.3 and 3.4.

The next two examples show that the conditions (R_1) and (R_2) (resp., (R_1)) in Theorem 3.4 (resp., Theorem 3.3) are not enough for the continuity (resp., the lower semicontinuity) of $\varphi(\cdot)$ at (D, A, c, b) if $QP(D, A, c, b)$ is embedded into the class of indefinite QP problems. In other words, they show that the results in Theorems 3.1 and 3.2 are different from the one in Theorem 3.4 (resp., Theorem 3.3).

EXAMPLE 3.6. Consider the problem $QP(D, A, c, b)$ where $n = m = 1$, $D = [0]$, $A = [1]$, $c = (1)$, $b = (0)$. We have $Sol(D, A, 0, 0) = \Delta(A, 0) = \{x \in R : x \geqslant 0\}$, $\varphi(D, A, c, b) = 0$. Besides, the system $Ax \geqslant b$ is regular. It is obvious that (R_1) and (R_2) hold. From Theorem 3.4 it follows that φ is continuous at (D, A, c, b) if $QP(D, A, c, b)$ is embedded into the class of convex QP problems. Meanwhile, by Theorem 2.1, φ is not continuous at (D, A, c, b) if $QP(D, A, c, b)$ is embedded into the class of indefinite QP problems.

EXAMPLE 3.7. Consider the same problem as in Example 3.6. We have seen that (R_1) is satisfied. By Theorem 3.3, φ is lower continuous at (D, A, c, b) if $QP(D, A, c, b)$ is embedded into the class of convex QP problems. Meanwhile, φ is not lower semicontinuous at (D, A, c, b) if $QP(D, A, c, b)$ is embedded into the class of indefinite QP problems. Indeed, consider the sequence $\{(D_k, A, c, b)\}$, where $D_k = [-\frac{1}{k}]$. For every k, we have $\varphi(D_k, A, c, b) = -\infty$. Therefore

$$\liminf_{k\to\infty} \varphi(D_k, A, c, b) = -\infty < 0 = \varphi(D, A, c, b).$$

This proves that φ is not lower semicontinuous at (D, A, c, b).

4. Concluding remarks

We have established some results on the continuity and semicontinuity of the optimal value function

$$(D, A, c, b) \mapsto \varphi(D, A, c, b) \tag{30}$$

of the problem $QP(D, A, c, b)$, where $D \in R_S^{n\times n}$ is an arbitrary symmetric matrix.

Continuity and Lipschitzian properties of the function

$$(c, b) \mapsto \varphi(D, A, c, b) \tag{31}$$

(the matrices $D,\ A$ are fixed) have been considered by several authors (see, e.g., [2, 3, 9, 13]). One referee of this paper noted that in [14] it has been proved that if D is positive semidefinite then (31) is a *piecewise quadratic* function. The referee also asked about piecewise quadratic properties of that when D is not assumed to be positive semidefinite. We are not able to consult with [14]. However, inspired by the observation of the referee, we have obtained a fact about piecewise quadratic property of the function in (31). For proving the property, we have to invoke some theorems on the continuity of the function (31) in [2], and also [9, Theorem 2].

Fixing a pair $(D, A) \in R_S^{n\times n} \times R^{m\times n}$, we define

$$\varphi(c, b) = \varphi(D, A, c, b), \quad Sol(c, b) = Sol(D, A, c, b).$$

The restriction of $\varphi(\cdot, \cdot)$ on the set $\mathcal{M} := \{(c, b) : Sol(c, b) \neq \emptyset\}$ is continuous (see [3, Theorems 1.1 and 1.2] and [2, Theorems 4.5.2]). The following fact seems to be new:

"In general, the function

$$\tilde{\varphi}(c, b) := \begin{cases} \varphi(c, b) & \textit{for all } (c, b) \in \mathcal{M} \\ +\infty & \textit{for all } (c, b) \in R^n \times R^m \setminus \mathcal{M} \end{cases}$$

is not piecewise quadratic in the sense of [13, Definition 10.20]."

Since the proof of this fact is quite long, it will be given in a subsequent paper. (See: G.M. Lee, N.N. Tam and N.D. Yen, On a class of optimal value functions in quadratic programming. Manuscript, June 2001. Submitted.)

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